Caringbah High School

Year 12 2023
Mathematics Extension 1
HSC Course
Assessment Task 4

General Instructions
- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 70

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **60 marks**
Attempt Questions 11-14
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**		
/10	/15	/15	/15	/15	/70	%

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10

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1. Which of the following vectors is parallel to the vector $\overrightarrow{OE} = -3\underset{\sim}{i} - 6\underset{\sim}{j}$?

(A) $\overrightarrow{OA} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$

(B) $\overrightarrow{OB} = -5\underset{\sim}{i} + 10\underset{\sim}{j}$

(C) $\overrightarrow{OC} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$

(D) $\overrightarrow{OD} = 4\underset{\sim}{i} - 8\underset{\sim}{j}$

2. Find the derivative of $2\tan^{-1}\dfrac{x}{4}$ with respect to x.

(A) $\dfrac{2}{4 + x^2}$

(B) $\dfrac{8}{16 + x^2}$

(C) $\dfrac{8}{1 + 4x^2}$

(D) $\dfrac{1}{2 + x^2}$

3. What is the equation of the horizontal asymptote of the function $y = \dfrac{2x}{4-x}$?

(A) $x = 4$

(B) $y = 2$

(C) $x = -2$

(D) $y = -2$

4. What is the range of the function $f(x) = \tan^{-1}(\sin x)$?

(A) $\left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

(B) $\left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

(C) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(D) $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

5. Find $\operatorname{proj}_{\underset{\sim}{w}} \underset{\sim}{v}$ given $\underset{\sim}{v} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{w} = 3\underset{\sim}{i} + \underset{\sim}{j}$

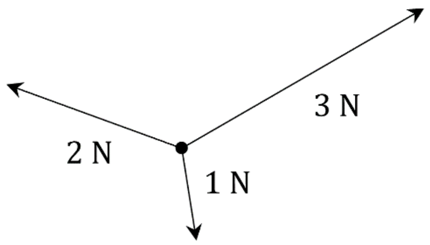
(A) $-\dfrac{11}{10}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

(B) $-\dfrac{11}{29}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

(C) $-\dfrac{11}{10}\left(-2\underset{\sim}{i} - 5\underset{\sim}{j}\right)$

(D) $-\dfrac{11}{29}\left(-2\underset{\sim}{i} - 5\underset{\sim}{j}\right)$

6. A particle is subject to forces of 1 N, 2 N, and 3 N in the directions shown.



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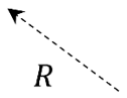
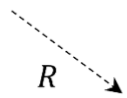
The sum of the forces acting on the particle is called the resultant force.

Which diagram best shows the resultant force, *R*?

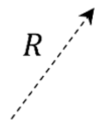
(A)



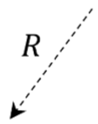
(B)



(C)



(D)



7. A sporting team needs 5 members, which must include at least two students from Year 11 and at least two students Year 12.

There are ten Year 11 students and fifteen Year 12 students available for selection.

In how many distinct ways can the team be chosen?

(A) 12

(B) 24

(C) 33 075

(D) 396 900

8. The number of solutions to the equation $(\sin^2 x - 1)(\tan^2 x - 1) = 0$ in the domain $\left[0,\, 2\pi\right]$ is

(A) 2

(B) 4

(C) 6

(D) 8

9. What is the coefficient of x^3 in the binomial expansion of $\left(x^2 - \frac{4}{x}\right)^9$?

(A) ${}^9C_4 4^5$

(B) ${}^9C_4 4^4$

(C) $-{}^9C_4 4^5$

(D) $-{}^9C_4 4^4$

10. Projectiles *A* and *B* are launched at the same time at velocity *V* and angle α. However projectile *A* is launched from a higher position. The two projectiles land in the same horizontal plane. Which of the following is always true?

(A) *A* and *B* will reach the ground at the same time.

(B) *A* and *B* will have the same range.

(C) *A* will reach its maximum height earlier that *B*.

(D) The maximum speed of *A* is greater than the maximum speed of *B*.

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

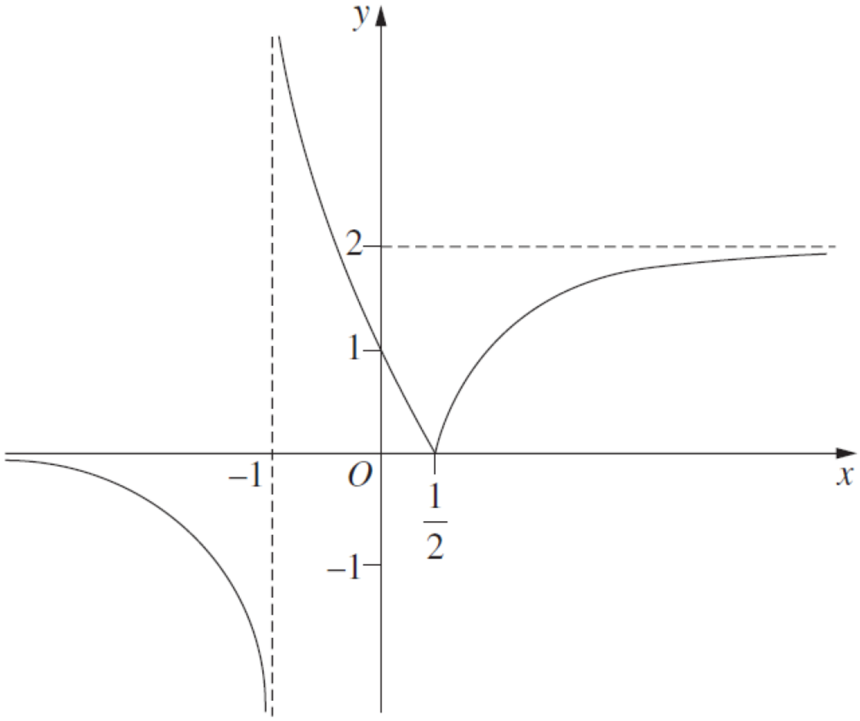
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the Cartesian equation of the parabola with parametric equations **2**

$$x = \frac{t}{2} + 5, \; y = t^2 - 1$$

(b) The diagram below shows the graph of $y = f(x)$



Draw sketches of the following on separate number planes:

(i) $y = f\left(|x|\right)$ **1**

(ii) $y^2 = f(x)$ **2**

Question 11 continues on page 7

(c) By using the principle of Mathematical Induction, prove that: **3**

$$6\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right) = n(n+1)(2n+1) \text{ for } n \geq 1.$$

(d) A, B and C are points defined by the position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} - 2\underset{\sim}{j}$ respectively

(i) Find $\overrightarrow{AB} \bullet \overrightarrow{BC}$ **2**

(ii) Hence, using part i, find the size of $\angle ABC$. **2**

(e) The letters A, E, I, O and U are vowels.

(i) How many arrangements of the word MACKENZIE are possible? **1**

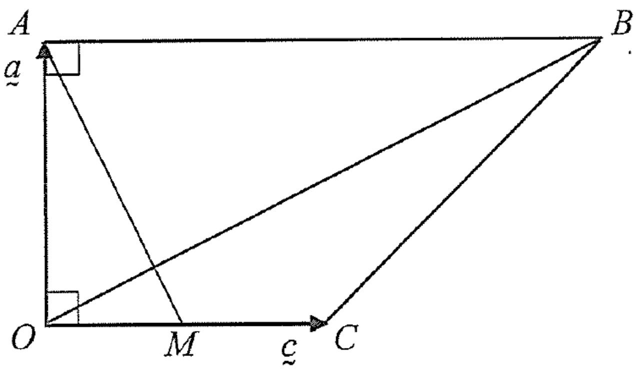
(ii) How many arrangements of the letters in the word MACKENZIE are possible if the vowels must occupy the 2nd, 3rd, 4th and 7th positions? **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) By writing equivalent equations in $t\left(\equiv \tan\frac{x}{2}\right)$, solve $5\sin x - 10\cos x = 2$ for $0^\circ \le x^\circ \le 360^\circ$. **3**

(b) **4**



NOT TO SCALE

In the diagram, $OABC$ is a trapezium with $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$,

$\angle COA = \angle BAO = 90^\circ$ and $OA = OC = \frac{1}{2}AB$. M is the midpoint of OC.

Use vector methods to show that $OB \perp AM$.

(c) The roots of $2x^3 + 6x + 3 = 0$ are α, β, γ.

Find the value of:

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(d) The area bound by the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (where a and b are constants), and the x-axis, is rotated about the x-axis. **4**

Find the volume of the solid of revolution formed.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A bowl of water heated to $100^\circ C$ is placed in a coolroom where the temperature is maintained at $-5^\circ C$. After *t* minutes, the temperature $T^\circ C$ of the water is changing so that $\frac{dT}{dt} = -k\left(T+5\right)$.

(i) Prove that $T = 105e^{-kt} - 5$ satisfies this equation. **1**

(ii) After 20 minutes, the temperature of the water has fallen to $40^\circ C$. How long, to the nearest minute, will the water need to be in the coolroom before the ice begins to form, (i.e. the temperature falls to $0^\circ C$) ? **2**

(b) Consider the following three expressions involving *n*, where *n* is a positive integer: $5^n + 3,\ 7^n + 5,\ 5^n + 7$

(i) By substituting values of *n*, show that $7^n + 5$ is the only one of these expressions which could be divisible by 6 for all positive integers *n*. **1**

(ii) Use Mathematical induction to show that the expression $7^n + 5$ is in fact divisible by 6 for all positive integers *n*. **2**

(c) **(i)** Use the substitution $x = \tan\theta$ to show that **3**

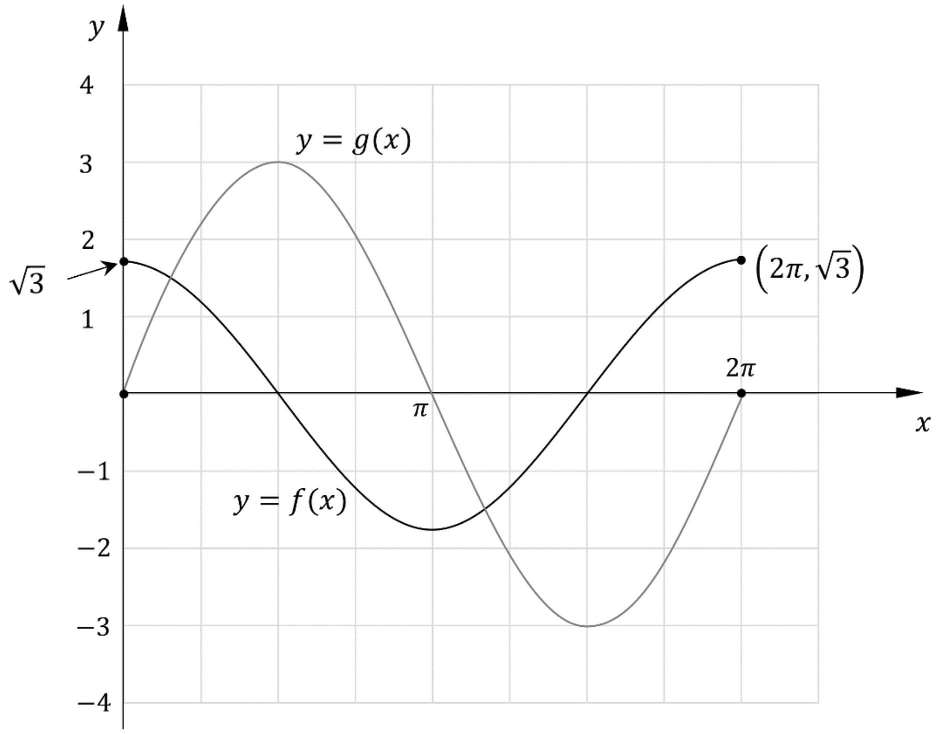
$$\int_0^1 \frac{4x^2}{(1+x^2)^3}\,dx = \int_0^{\frac{\pi}{4}} \sin^2 2\theta\, d\theta.$$

(ii) Hence, find in simplest form the exact value of **3**

$$\int_0^1 \frac{4x^2}{(1+x^2)^3}\,dx$$

Question 13 continues of the next page.

(d) The graphs of the trigonometric curves $y = f(x)$ and $y = g(x)$ are shown below. **3**



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Find the equation of the curve $y = f(x) + g(x)$, expressed in the form $y = R\cos(x - \alpha)$, where α is in radians.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) At the same instant two stones A and B are projected from a point O. Stone A is projected with speed $V\text{ ms}^{-1}$ at an angle α above the horizontal and stone B is projected with speed $2V\text{ ms}^{-1}$ at an angle 2α above the horizontal, where $0 < \alpha < \frac{\pi}{4}$. The two stones move above the horizontal ground in the same vertical plane under gravity where the acceleration due to gravity is $g\text{ ms}^{-2}$, and hit the ground at the same point X. At time t seconds the position vectors of the two stones relative to O are respectively

$$\underset{\sim}{r}_A(t) = (Vt\cos\alpha)\underset{\sim}{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\underset{\sim}{j} \text{ and}$$

$$\underset{\sim}{r}_B(t) = (2Vt\cos 2\alpha)\underset{\sim}{i} + \left(2Vt\sin 2\alpha - \frac{1}{2}gt^2\right)\underset{\sim}{j}.$$

(i) Show that stone A has a horizontal range $R_A = \dfrac{V^2\sin 2\alpha}{g}$ and state the corresponding expression for the horizontal range R_B of stone B. **2**

(ii) Show that $\cos 2\alpha = \dfrac{1}{8}$ and hence find in simplest form the value of $\cos\alpha$. **2**

(iii) If T_A and T_B are respectively the times of flight of stone A and stone B, find in simplest exact form the ratio $\dfrac{T_B}{T_A}$. **2**

(b) Solve the equation $\sin 3x - \sin x + \cos 2x = 0$ for $0 \le x \le 2\pi$ **4**

Question 14 continues of the next page.

(c) Let p and q be positive integers with $p \leq q$.

(i) Use the binomial theorem to expand $(1+x)^{p+q}$, and hence write down the term of $\dfrac{(1+x)^{p+q}}{x^q}$ which is independent of x. **2**

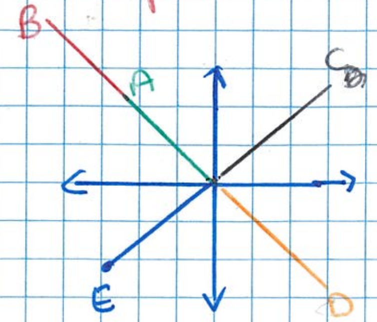
(ii) Given that $\dfrac{(1+x)^{p+q}}{x^q} = (1+x)^p \left(1+\dfrac{1}{x}\right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for **3**

$$1+\binom{p}{1}\binom{q}{1}+\binom{p}{2}\binom{q}{2}+\dots+\binom{p}{p}\binom{q}{p}.$$

End of Examination.

Multiple Choice

Q1.



$\therefore C$

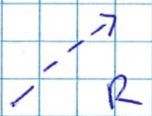
Q2 $\dfrac{8}{16+x^2}$ $\therefore B$

Q3 $y = -2$ $\therefore D$

Q4 $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ $\therefore B$

Q5 $proj_{\underset{\sim}{w}} \underset{\sim}{v} = \frac{-11}{10}(3\underset{\sim}{i} + \underset{\sim}{j})$ $\therefore A$

Q6

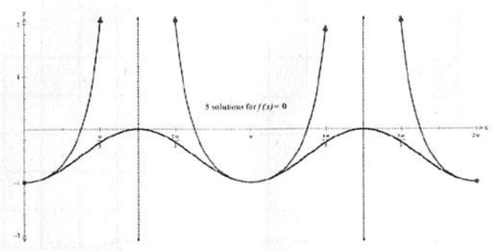


$\therefore C$

Q7 ${}^{10}C_2 \times {}^{15}C_3 + {}^{10}C_3 \times {}^{15}C_2$

$= 33075$ $\therefore C$

Q8



Asymptote values (solutions for sin) cannot be valid for tan.
4 solutions

$\therefore B$

Q9 $\left(\frac{9}{4}\right)\cdot(-4)^5 = -\left(\frac{9}{4}\right)4^5$ $\therefore C$

Q10 Max speed of A is greater than max speed of B. $\therefore D$

Q11

a. $x = \frac{t}{2} + 5$ $\qquad y = t^2 - 1$

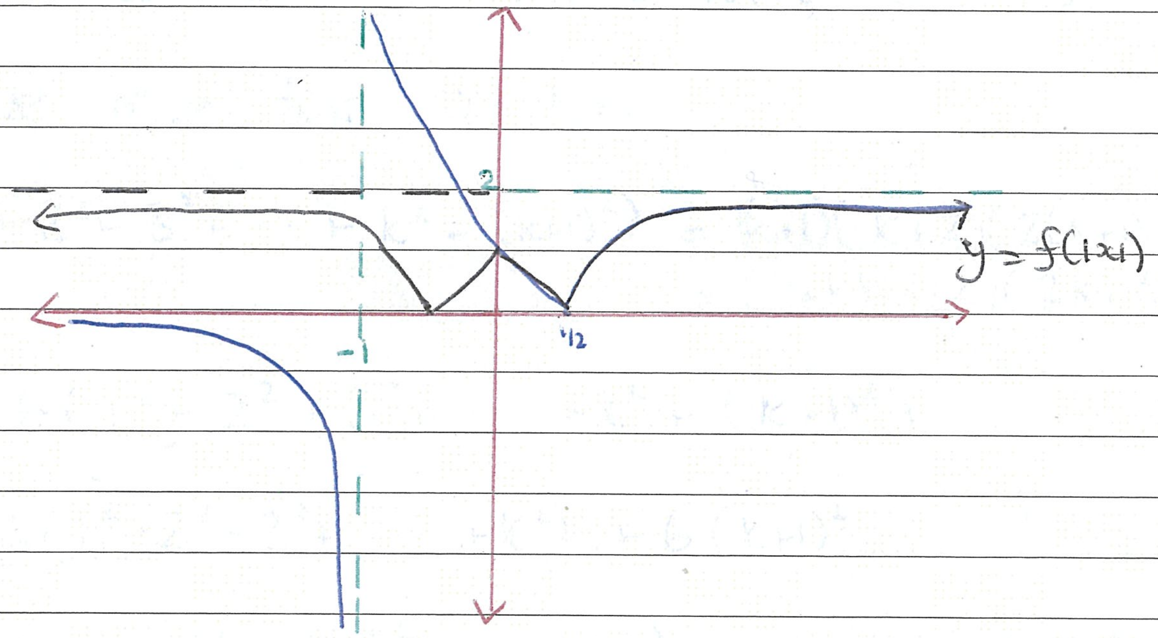
$2x = t + 10$

$t = 2x - 10 \Rightarrow y = (2x-10)^2 - 1$

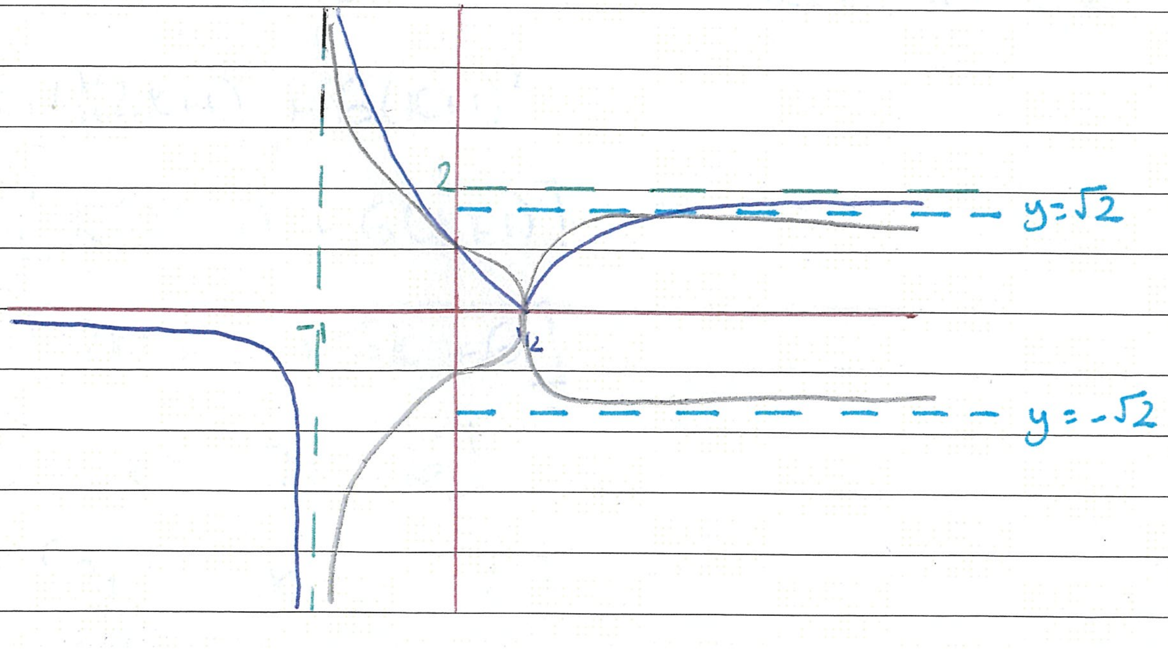
$= 4x^2 - 40x + 100 - 1$

$y = 4x^2 - 40x + 99$

b i



ii



c. $6(1^2+2^2+3^2+\dots+n^2) = n(n+1)(2n+1)$

for $n \geq 1$.

① Prove for $n=1$

$$6(1^2) = 1(1+1)(2(1)+1)$$
$$6 = 1(2)(3)$$
$$= 6 \quad ✓ \quad \therefore \text{ true for } n=1$$

② assume true for $n=k$ for some int k.

$$6(1^2+2^2+3^2+\dots+k^2) = k(k+1)(2k+1)$$

③ Prove true for $n=k+1$

$$6(1^2+2^2+3^2+\dots+k^2+(k+1)^2) = (k+1)(k+2)(2(k+1)+1)$$
$$= (k+1)(k+2)(2k+3)$$

$$LHS = 6(1^2+2^2+3^2+\dots+k^2+(k+1)^2)$$

$$= 6(1^2+2^2+3^2+\dots+k^2) + 6(k+1)^2$$

From ② $6(1^2+2^2+3^2+\dots+k^2) = k(k+1)(2k+1)$

$$\therefore = k(k+1)(2k+1) + 6(k+1)^2$$

$$= (k+1)[k(2k+1) + 6(k+1)]$$

$$= (k+1)[2k^2+k+6k+6]$$

$$= (k+1)[2k^2+7k+6]$$

$$= (k+1)[2k^2+4k+3k+6]$$

$= (k+1)\left[2k(k+2) + 3(k+2)\right]$

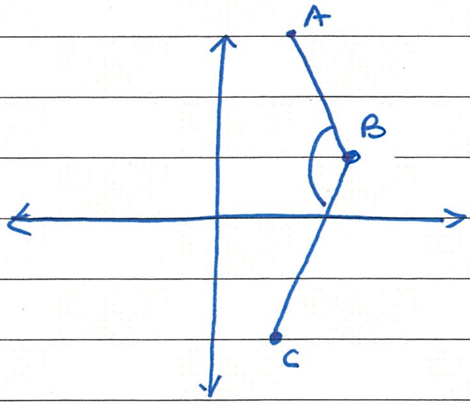
$= (k+1)(k+2)(2k+3)$

$=$ RHS

$\therefore$ If the statement is true for $n=1$, it is true for $n=2$ & if the statement is true for $n=2$ it is true for $n=3$ $\therefore$ by the process of mathematical induction it is true for all $n \geq 1$.

d.i $\overrightarrow{AB} = 2\underset{\sim}{i} + \underset{\sim}{j} - (\underset{\sim}{i} + 3\underset{\sim}{j})$

$= \underset{\sim}{i} - 2\underset{\sim}{j}$



$\overrightarrow{BC}$ $\underset{\sim}{i} - 2\underset{\sim}{j} - (2\underset{\sim}{i} + \underset{\sim}{j})$

$= -\underset{\sim}{i} - 3\underset{\sim}{j}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 1 \times -1 + (-2 \times -3)$

$= 5$

ii $|\overrightarrow{AB}| = \sqrt{1^2 + (-2)^2}$

$= \sqrt{5}$

$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (-3)^2}$

$= \sqrt{10}$

$\cos\theta = \dfrac{-5}{\sqrt{5} \cdot \sqrt{10}}$

$= \dfrac{-5}{5\sqrt{2}}$

$= \dfrac{-1}{\sqrt{2}}$

$\Rightarrow$ we use -5 as we need $\overrightarrow{AB} \cdot \overrightarrow{CB}$ as $\angle ABC$ is obtuse

$\therefore \theta = \dfrac{3\pi}{4}$

11e i M a c k e n z i e

total arrangements $= \frac{9!}{2!}$

$= 181\,440$

ii 5 4 3 2 4 3 1 2 1

✓ ✓ ✓ ✓

restricted arrangements $= \frac{5! \times 4!}{2!}$

$= 1440$

Q12a. $5\sin x - 10\cos x = 2$

$\cancel{LHS}\ 2 = 5\sin x - 10\cos x$

$2 = 5 \cdot \frac{2t}{1+t^2} - 10 \cdot \frac{(1-t^2)}{1+t^2}$

$2 = \frac{10t}{1+t^2} - \frac{10+10t^2}{1+t^2}$

$2 = \frac{10t^2+10t-10}{1+t^2}$

$2+2t^2 = 10t^2+10t-10$

$8t^2+10t-12=0$

$4t^2+5t-6=0$

$t = \frac{-5 \pm \sqrt{5^2-4\times4\times-6}}{2\times4}$

$= \frac{-5\pm\sqrt{121}}{8}$

$= \frac{3}{4}, -2$

$\therefore \tan\frac{1}{2}x = \frac{3}{4}, \qquad \tan\frac{1}{2}x = -2$

$0^\circ \leq \frac{1}{2}x \leq 180^\circ$

$\therefore \frac{1}{2}x = 36^\circ52', \qquad \frac{1}{2}x = 116^\circ34'$

$\therefore x = 73^\circ44' \qquad x = 233^\circ8'$

b) $2x^3 + 6x + 3 = 0$

$\therefore a = 2$ $\quad \alpha+\beta+\gamma = -0/2 = 0$

$b = 0$

$c = 6$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma = 6/2 = 3$

$d = 3$

$\alpha\beta\gamma = -3/2$

i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{3}{-3/2}$

$= -2$

ii $\alpha^2+\beta^2+\gamma^2$: $\cancel{(\alpha+\beta+\gamma)^2 =}$

$(\alpha+\beta+\gamma)^2 = \alpha^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \beta^2 + \gamma^2$

$= \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0^2 - 2(3)$

$= -6$

c. $\angle COA = \angle BAO = 90^\circ$

$\therefore \underbrace{\underset{\sim}{a} \cdot \underset{\sim}{c}}{} = 0$ & $OC \parallel AB$.

Then $OC = \frac{1}{2} AB \Rightarrow \overrightarrow{AB} = 2\underset{\sim}{c}$

& $\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{c}$

$\overrightarrow{OB} \cdot \overrightarrow{AM} = (\underset{\sim}{a} + 2\underset{\sim}{c}) \cdot (-\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c})$

$= -\underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{c} \cdot \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{a} \cdot \underset{\sim}{c} - 2\underset{\sim}{a} \cdot \underset{\sim}{c}$

$= -OA^2 + OC^2 - \frac{3}{2}\underbrace{\underset{\sim}{a} \cdot \underset{\sim}{c}}_{\text{from line 2.}}$

$= \underbrace{0 + 0}_{\text{since } OA = OC} - 0$

$= 0$

$\therefore OB \perp AM$

d. $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Find x intercepts. (let $y = 0$)

$\therefore 0 = \frac{b}{a}\sqrt{a^2 - x^2}$

$0 = \sqrt{a^2 - x^2}$

$x^2 = a^2$

$x = \pm a$.

let $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$

$f(-x) = \frac{b}{a}\sqrt{a^2-(-x)^2}$

$= \frac{b}{a}\sqrt{a^2-x^2}$ $\therefore$ even

$\therefore$ symmetrical about y-axis.

$V = \pi \int_{-a}^{a} y^2 \, dx$

$= \pi \int_{-a}^{a} \left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2 dx$

$= \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\, dx$

$V = \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$

$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3} - 0\right]$

$= \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right)$

$V = \frac{4\pi}{3} b^2 a \ u^3.$

Q 13.

i $T = 105e^{-kt} - 5$

$\frac{dT}{dt} = -105ke^{-kt}$

$= -k(105e^{-kt})$

$= -k(105e^{-kt} + 5 - 5)$

$= -k(T + 5)$

ii when $t = 20$; $T = 40$

$\therefore 40 = 105e^{-20k} - 5$

$45 = 105e^{-20k}$

$\frac{3}{7} = e^{-20k}$

$\ln\frac{3}{7} = -20k$

$k = \ln\left(\frac{3}{7}\right) \div -20$

find t when $T = 0$

$0 = 105e^{-kt} - 5$

$\frac{5}{105} = e^{-kt}$

$\ln\left(\frac{5}{105}\right) = -kt$

$\therefore \quad t = \ln\left(\frac{5}{105}\right) \div -k$

$= 71.864 \ldots$

$\doteqdot 72$ minutes.

bi let $n=1$ let $n=2$

$\therefore \quad 5^n + 3 = 8$ ✗ $\quad 5 = 28$ ✗

$7^n + 5 = 12$ ✓ $\quad = 54$ ✓

$5^n + 7 = 12$ ✓ $\quad = 32$ ✗

ii ① ~~pr~~ shown true for $n=1$ above

② assume true for $n=k$, for some integer k

$\therefore \quad 7^k + 5 = 6p$ for some integer p

③ prove true for $n = k+1$

$7^{k+1} + 5 = 6q$ for some integer q

from ② $\quad 7^k + 5 = 6p$

$7^k = 6p - 5$

$7^{k+1} + 5 = \cancel{\text{R}}$ LHS

$= 7^k \cdot 7^1 + 5$

$= (6p-5)\cdot 7 + 5$

$= 6\cdot 7\cdot p - 35 + 5$

$= 6\times 7\times p - 30$

$= 6(7p-5)$

$= 6q$ where $q = 7p-5$

$\therefore$ $7^{k+1}+5$ is divisible by 6. for $n=k+1$

Since the statement is true for $n=1$, it is true for $n=2$. Since it is true for $n=2$ it is true for $n=3$. $\therefore$ by the process of mathematical induction it is true for all integers $n \geq 1$

13c i

$$\int_0^1 \frac{4x^2}{(1+x^2)^3}\,dx$$

let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{(1+\tan^2\theta)^3} \times \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{(\sec^2\theta)^3} \times \sec^2\theta \, d\theta$$

$1 = \tan\theta$

$\therefore \theta = \pi/4$

$0 = \tan\theta$

$\therefore \theta = 0$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{(\sec^2\theta)^2}\, d\theta$$

$$= \int_0^{\pi/4} 4\cdot \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/4} 4 \cdot \frac{\sin^2\theta}{\cos^2\theta} \cdot (\cos^2\theta)^2 \, d\theta$$

$$= \int_0^{\pi/4} 4 \cdot \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/4} (2\sin\theta\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi/4} (\sin 2\theta)^2 \, d\theta$$

$$= \int_0^{\pi/4} \sin^2 2\theta \, d\theta$$

ii $\int_0^1 \frac{4x^2}{(1+x^2)^3} \, dx = \int_0^{\pi/4} \sin^2 2\theta \, d\theta$ from part (i)

$$= \int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\theta - \frac{1}{4}\sin\pi\right) - \left(0 - \frac{1}{4}\sin 0\right)\right]$$

$$= \frac{1}{2}\left[(\theta - 0) - (0 - 0)\right]$$

$$= \theta$$

13d. $f(x) = \sqrt{3}\cos x$ $\qquad$ $g(x) = 3\sin x$

$$y = 3\sin x + \sqrt{3}\cos x$$

$$y = R\cos(x - a)$$

$$R = \sqrt{(3)^2 + (\sqrt{3})^2}$$
$$= \sqrt{12}$$
$$= 2\sqrt{3}$$

$$2\sqrt{3}\cos(x - \alpha) = 2\sqrt{3}\cos x\cos\alpha + 2\sqrt{3}\sin x\sin\alpha$$

$$\therefore\ 3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\cos x\cos\alpha + 2\sqrt{3}\sin x\sin\alpha$$

$\therefore\ 3 = 2\sqrt{3}\sin\alpha$, $\sqrt{3} = 2\sqrt{3}\cos\alpha$

$\sin\alpha = \dfrac{3}{2\sqrt{3}}$ $\qquad$ $\cos\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}}$

$$\cos\alpha = \frac{1}{2}$$

$$\therefore\ \alpha = \frac{\pi}{3}$$

$$\therefore\ y = 2\sqrt{3}\cos\left(x - \frac{\pi}{3}\right)$$

Q14a

i $\underset{\sim}{R_A}(t) = (Vt\cos\alpha)\underset{\sim}{i} + (Vt\sin\alpha - \frac{1}{2}gt^2)\underset{\sim}{j}$

$y = 0, \quad t \neq 0 \qquad \therefore \quad Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$$\frac{1}{2}gt\left(\frac{2V\sin\alpha}{g} - t\right) = 0$$

$$\therefore \quad t = \frac{2V\sin\alpha}{g}$$

$x = Vt\cos\alpha$

$= V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$

$= \frac{2V^2\sin\alpha\cos\alpha}{g}$

$\underset{\sim}{R_A} = \frac{V^2\sin 2\alpha}{g}$

$\cancel{\underset{\sim}{R_B} = V\left(\frac{2V\sin\alpha}{g}\right)}$

$\underset{\sim}{R_B} = \frac{(2V)^2 \cdot \sin 2(2\alpha)}{g}$

$= \frac{4V^2\sin 4\alpha}{g}$

ii since both stones are fired from O & land at the same point $R_A = R_B$

$$\therefore \frac{V^2 \sin 2\alpha}{g} = \frac{4V^2 \sin 4\alpha}{g}$$

$$\sin 2\alpha = 4 \sin 4\alpha$$

$$= \cancel{4}(2 \sin 2\cancel{\theta}\alpha \cos \cancel{2\theta}\alpha)$$

$$= 8 \sin 2\cancel{\theta}\alpha \cos 2\alpha$$

as $\alpha \neq 0$ $\quad 1 = 8 \cos 2\alpha$

$$\cos 2\alpha = \frac{1}{8}$$

$$2\cos^2\alpha - 1 = \frac{1}{8}$$

$$2\cos^2\alpha = \frac{9}{8}$$

$$\cos^2\alpha = \frac{9}{16} \qquad 0 \leq \alpha \leq \frac{\pi}{4}$$

$$\therefore \cos\alpha > 0$$

$$\therefore \cos\alpha = \frac{3}{4}$$

iii $T_A = \frac{2V\sin\alpha}{g}$ & $T_B = \frac{2(2V)\sin 2\alpha}{g}$

$$= \frac{4V\sin 2\alpha}{g}$$

$$= \frac{8V\sin\alpha\cos\alpha}{g}$$

$$T_B = 4\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$$T_B = 4 \times T_A \times \cos\alpha$$

$$\therefore \frac{T_B}{T_A} = 4\cos\alpha$$

$$= 4 \times \frac{3}{4}$$

$$\frac{T_B}{T_A} = 3$$

b. $\sin 3x - \sin x + \cos 2x$

$= \sin(2x + x) - \sin x + \cos 2x$

$= \sin 2x \cos x + \cos 2x \sin x - \sin x + \cos 2x$

$= 2\sin x \cos^2 x + (2\cos^2 x - 1)\sin x - \sin x + \cos 2x$

$= 2\sin x \cos^2 x + 2\sin x \cos^2 x - \sin x - \sin x + \cos 2x$

$= 4\sin x \cos^2 x - 2\sin x + \cos 2x$

$= 4\sin x(1-\sin^2 x) - 2\sin x + 1 - 2\sin^2 x$

$= 4\sin x - 4\sin^3 x - 2\sin x + 1 - 2\sin^2 x$

$= -4\sin^3 x - 2\sin^2 x + 2\sin x + 1$

$= -2\sin^2 x(2\sin x + 1) + 1(2\sin x + 1)$

$= (2\sin x + 1)(1 - 2\sin^2 x)$

$\therefore (2\sin x + 1)(1 - 2\sin^2 x) = 0$

$\therefore 2\sin x + 1 = 0, \quad 1 - 2\sin^2 x = 0$

$2\sin x = -1, \quad 2\sin^2 x = 1$

$\sin x = -\frac{1}{2} \qquad \sin^2 x = \frac{1}{2}$

$x = \frac{11\pi}{6}, \frac{7\pi}{6} \qquad \sin x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

c) ~~The probability of answering the first a question correctly is p.~~

~~$\therefore$ the probability of answering~~

C.i $(1+x)^{p+q} = \sum_{k=0}^{p+q} {}^{p+q}C_k\, x^k$

$= {}^{p+q}C_0\, x^0 + {}^{p+q}C_1\, x^1 + \dots\dots + {}^{p+q}C_p\, x^p + \dots\dots + {}^{p+q}C_q\, x^q$

$+ \dots\dots + {}^{p+q}C_{p+q}\, x^{p+q}$

If $p=q$ this term would not exist.

$\frac{(1+x)^{p+q}}{x^q} = {}^{p+q}C_0 \frac{x^0}{x^q} + {}^{p+q}C_1 \frac{x^1}{x^q} + \dots\dots + \boxed{{}^{p+q}C_p \frac{x^p}{x^q}} +$

$\dots\dots + \boxed{{}^{p+q}C_q \frac{x^q}{x^q}} + \dots\dots + {}^{p+q}C_{p+q} \frac{x^{p+q}}{x^q}$

this is the term independent of x

$\therefore$ term independent of x is ${}^{p+q}C_q$

.i Given $\frac{(1+x)^{p+q}}{x^q} = (1+x)^p \left(1+\frac{1}{x}\right)^q$

$(1+x)^p = \sum_{k=0}^{p} {}^pC_k\, x^k$

$= {}^pC_0\, x^0 + {}^pC_1\, x^1 + {}^pC_2\, x^2 + \dots\dots + {}^pC_p\, x^p$

$\left(1+\frac{1}{x}\right)^q = \sum_{k=0}^{q} \frac{{}^qC_0}{x^0} + \frac{{}^qC_1}{x^1} + \frac{{}^qC_2}{x^2} + \dots\dots + \frac{{}^qC_p}{x^p} + \dots\dots \frac{{}^qC_q}{x^q}$

$\therefore (1+x)^p \cdot \left(1+\frac{1}{x}\right)^q = \frac{{}^pC_0\, x^0 \times {}^qC_0}{x^0} + \dots\dots + \frac{{}^pC_1\, x^1 \times {}^qC_1}{x^1}$

$+ \dots\dots + \frac{{}^pC_2\, x^2 \times {}^qC_2}{x^2} + \dots\dots$

$+ \frac{{}^pC_p\, x^p \times {}^qC_p}{x^p} + \dots\dots$

$$= {}^pC_0 \cdot {}^qC_0 + {}^pC_1\,{}^qC_1 + {}^pC_2\,{}^qC_2 + \dots + {}^pC_p\,{}^qC_p + \text{terms not ind. of } x.$$

$\therefore$ by equating terms independent of x on LHS + RHS

$${}^{p+q}C_q = 1 + {}^pC_1\,{}^qC_1 + {}^pC_2\,{}^qC_2 + \dots + {}^pC_p\,{}^qC_p$$

$$\boxed{{}^{p+q}C_q} = 1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \dots + \binom{p}{p}\binom{q}{p}$$

this is the simpler expression